

# Double-Star Decomposition of Regular Graphs <sup>\*†</sup>

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## Abstract

A tree containing exactly two non-pendant vertices is called a double-star. A double-star with degree sequence  $(k_1 + 1, k_2 + 1, 1, \dots, 1)$  is denoted by  $S_{k_1, k_2}$ . We study the edge-decomposition of regular graphs into double-stars. It was proved that every double-star of size  $k$  decomposes every  $2k$ -regular graph. In this paper, we extend this result to  $(2k + 1)$ -regular graphs, by showing that every  $(2k + 1)$ -regular graph containing two disjoint perfect matchings is decomposed into  $S_{k_1, k_2}$  and  $S_{k_1 - 1, k_2}$ , for all positive integers  $k_1$  and  $k_2$  such that  $k_1 + k_2 = k$ .

## 1 Introduction

Let  $G = (V(G), E(G))$  be a graph and  $v \in V(G)$ . We denote the set of all neighbors of  $v$  by  $N(v)$ . The degree of a vertex  $v$  in  $G$  is denoted by  $d_G(v)$  (for abbreviation  $d(v)$ ). By *size* and *order* of  $G$  we mean  $|E(G)|$  and  $|V(G)|$ , respectively. Let  $X \subseteq V(G)$ , then the *induced subgraph* with vertex set  $X$  is denoted by  $G[X]$ . A subset  $M \subseteq E(G)$  is called a *matching* if no two edges of  $M$  are incident. A matching  $M$  is called a *perfect matching*, if every vertex of  $G$  is incident with an edge of  $M$ . A *factor* of  $G$  is a spanning subgraph

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of  $G$ . A subgraph  $H$  is called an  $r$ -factor if  $H$  is a factor of  $G$  and  $d_H(v) = r$ , for every  $v \in V(G)$ .

If  $d(v) = 1$ , then  $v$  is called a *pendant vertex*. A tree containing exactly two non-pendant vertices is called a *double-star*. A double-star with degree sequence  $(k_1 + 1, k_2 + 1, 1, \dots, 1)$  is denoted by  $S_{k_1, k_2}$ . Suppose that  $u_1, u_2 \in V(S_{k_1, k_2})$  and  $d(u_i) = k_i + 1$ , for  $i = 1, 2$ . Let  $X$  and  $Y$  be the set of all pendant vertices adjacent to  $u_1$  and  $u_2$ , respectively. Then we say that  $S_{k_1, k_2}$  is a double-star with pendant sets  $X$  and  $Y$ . Also,  $e = u_1 u_2$  is called the *central edge* of the double-star.

Let  $G$  and  $H$  be two graphs. The *Cartesian product* of  $G$  and  $H$  is denoted by  $G \square H$  and is a graph with vertex set  $\{(u, v) : u \in G, v \in H\}$  and two vertices  $(u, v)$  and  $(x, y)$  are adjacent if and only if  $u = x$  and  $v$  is adjacent to  $y$ , or  $v = y$  and  $u$  is adjacent to  $x$ .

For a graph  $H$  the graph  $G$  has an  $H$ -decomposition, if all edges of  $G$  can be partitioned into subgraphs isomorphic to  $H$ . Also, we say that  $G$  has an  $\{H_1, \dots, H_t\}$ -decomposition if all edges of  $G$  can be partitioned into subgraphs, each of them isomorphic to some  $H_i$ , for  $1 \leq i \leq t$ . If  $G$  has an  $H$ -decomposition, we say that  $G$  is  $H$ -decomposable. A graph  $G$  is  $k$ -factorable if it can be decomposed into  $k$ -factors.

Let  $G$  be a directed graph and  $v \in V(G)$ . We define  $N^+(v) = \{u \in V(G) : (v, u) \in E(G)\}$ , where  $(v, u)$  denotes the edge from  $v$  to  $u$ . By *out-degree* of  $v$  we mean  $|N^+(v)|$  and denote it by  $d_G^+(v)$ . Similarly, we define  $N^-(v) = \{u \in V(G) : (u, v) \in E(G)\}$  and denote  $|N^-(v)|$  by  $d_G^-(v)$ . An orientation  $O$  is called *Eulerian* if  $d_G^+(v) = d_G^-(v)$ , for every  $v \in V(G)$ . A  $k$ -orientation is an orientation such that  $d_G^+(v) = k$ , for every  $v \in V(G)$ . A  $\{k_1, \dots, k_t\}$ -orientation is an orientation such that for every  $v \in V(G)$ ,  $d_G^+(v) = k_i$ , for some  $1 \leq i \leq t$ . In 1979, K tzig conjectured that every  $(2k+1)$ -regular graph is decomposed into  $S_{k, k}$  if and only if it has a perfect matching. Jaeger, Payan and Kouider in 1983 proved this conjecture, see [5]. El-Zanati et al. proved that every  $2k$ -regular graph containing a perfect matching is  $S_{k, k-1}$ -decomposable, see [3]. The following interesting conjecture was proposed by Ringel, see [7].

**Conjecture 1.** *Every tree of size  $k$  decomposes the complete graph  $K_{2k+1}$ .*

A broadening of Ringel's conjecture is due to Graham and H ggkvist.

**Conjecture 2.** *Every tree of size  $k$  decomposes every  $2k$ -regular graph.*

El-Zanati et al. proved the following theorem in [3].

**Theorem 1.** *Every double-star of size  $k$  decomposes every  $2k$ -regular graph.*

Jacobson et al. in 1991 proposed the following conjecture about the tree decomposition of regular bipartite graphs, see [4].

**Conjecture 3.** *Let  $T$  be a tree of size  $r$ . Then every  $r$ -regular bipartite graph is  $T$ -decomposable.*

They proved that the conjecture holds for double-stars. In this paper, we study double-star decomposition of regular graphs. First, we prove some results about the double-star decomposition of regular bipartite graphs. We present a short proof for Conjecture 3, when  $T$  is a double-star. Then we study the double-star decomposition of  $(2k+1)$ -regular graphs. As a straight result of Theorem 1, we prove that every  $(2k+1)$ -regular graph containing a 2-factor, has  $S$ -decomposition, where  $S = \{S_{k_1, k_2}, S_{k_1-1, k_2}, S_{k_1, k_2-1}, S_{k_1-1, k_2-1}\}$  for any double-star  $S_{k_1, k_2}$  of size  $k+1$ . Then we present a theorem which indicates that every  $(2k+1)$ -regular graph containing two disjoint perfect matchings is  $\{S_{k_1, k_2}, S_{k_1-1, k_2}\}$ -decomposable, for any double-star  $S_{k_1, k_2}$  of size  $k+1$ . Also, we prove that every triangle-free  $(2k+1)$ -regular graph containing a 2-factor, is  $\{S_{k_1, k_2}, S_{k_1+1, k_2}\}$ -decomposable, for any double-star  $S_{k_1, k_2}$  of size  $k$ .

## 2 Double-Star Decomposition of Regular Bipartite Graphs

In this section, we prove some results about the double-star decomposition of regular bipartite graphs. The following theorem was proved in [4]. We present a short proof for this result.

**Theorem 2.** *For  $r \geq 3$ , let  $G$  be an  $r$ -regular bipartite graph. Then every double-star of size  $r$  decomposes  $G$ .*

*Proof.* Let  $A$  and  $B$  be two parts of  $G$ . Then König Theorem [1, Theorem 2.2] implies that  $G$  has a 1-factorization with 1-factors  $M_1, \dots, M_r$ . Suppose that  $S_{k_1, k_2}$  is a double-star of size  $r$ . Now, let  $G_1$  and  $G_2$  be two induced subgraphs of  $G$  with the edges  $M_1 \cup M_2 \cup \dots \cup M_{k_1}$  and  $M_{k_1+1} \cup \dots \cup M_{r-1}$ , respectively. Suppose that  $e = u_1 u_2 \in M_r$ , where  $u_1 \in A$  and  $u_2 \in B$ . Now, define  $S_e$  the double-star containing the central edge  $e$ ,  $E_1(u_1)$  and  $E_2(u_2)$ , where  $E_i(u_i)$  is the set of all edges incident with  $u_i$  in  $G_i$ . Clearly,  $S_e$  is isomorphic to  $S_{k_1, k_2}$ . On the other hand,  $S_e$  and  $S_{e'}$  are edge disjoint, for every two distinct edges  $e, e' \in M_r$ . Hence,  $E(G) = \cup_{e \in M_1} S_e$  and this completes the proof.  $\square$

Now, we have the following corollaries.

**Corollary 1.** *Let  $r, s \geq 3$  be positive integers and  $s \mid r$ . Then every  $r$ -regular bipartite graph can be decomposed into any double-star of size  $s$ .*

*Proof.* Let  $r = sk$  and  $S_{k_1, k_2}$  be a double-star of size  $s$ . Since  $G$  is 1-factorable,  $G$  can be decomposed into spanning subgraphs  $G_1, \dots, G_k$ , each of them is  $s$ -regular. Now, Theorem 2 implies that each  $G_i$  can be decomposed into  $S_{k_1, k_2}$  and this completes the proof.  $\square$

**Corollary 2.** *Let  $r, s, k$  and  $t$  be positive integers such that  $r = sk + t$  and  $r, s, t \geq 3$ . Moreover, suppose that  $S_1$  and  $S_2$  be two double-stars of size  $s$  and  $t$ , respectively. Then every  $r$ -regular bipartite graph  $G$  is  $\{S_1, S_2\}$ -decomposable.*

*Proof.* Similar to the proof of the previous corollary,  $G$  is decomposed into  $G_1, \dots, G_{k+1}$  where  $G_1, \dots, G_k$  are  $s$ -regular and  $G_{k+1}$  is  $t$ -regular. Now, Theorem 2 implies that  $G_1, \dots, G_k$  and  $G_{k+1}$  can be decomposed into  $S_1$  and  $S_2$ , respectively. This completes the proof.  $\square$

Another generalization of Theorem 2 is as follows.

**Theorem 3.** *Let  $r \geq 3$  be an integer and  $G = (A, B)$  be a bipartite graph such that for every  $v \in V(G)$ ,  $r \mid d(v)$ . Then every double-star of size  $r$  decomposes  $G$ .*

*Proof.* Let  $v \in A$  and  $d(v) = rk$ , for some positive integer  $k$  and  $S$  be a double-star of size  $r$ . Suppose that  $N(v) = \{u_1, \dots, u_{rk}\}$ . Let  $G'$  be the graph obtained from  $G$  by removing  $v$  and adding  $v_1, \dots, v_k$  to  $A$ . For  $i = 1, \dots, k$ , join  $v_i$  to every vertex of the set  $\{u_{(i-1)r+1}, \dots, u_{ir}\}$ . It is not hard to see that if  $G'$  is  $S$ -decomposable, then  $G$  is  $S$ -decomposable, too.

By repeating this procedure for all vertices of  $G$ , one can obtain an  $r$ -regular bipartite graph, say  $H$ . Now, Theorem 2 implies that  $H$  is  $S$ -decomposable and hence  $G$  is  $S$ -decomposable.  $\square$

### 3 Double-Star Decomposition of Odd Regular Graphs

In this section, we use the following structure which was used in [4]. Let  $G$  be a  $2k$ -regular graph. Then Petersen Theorem [1, Theorem 3.1] implies that  $G$  is 2-factorable. Let  $F$  be a 2-factor of  $G$  with cycles  $C_1, \dots, C_l$ . Note that  $G \setminus F$  has an Eulerian orientation. Also, orient  $C_i$  clockwise, for  $i = 1, \dots, l$ , to obtain an Eulerian orientation of  $G$ . We define  $G_{C_i}$

as the subgraph of  $G$  with the edge set  $E = \{(u, v) : u \in V(C_i)\}$ . Clearly,  $\{G_{C_1}, \dots, G_{C_l}\}$  partitions  $E(G)$ . So, if we show that each  $G_{C_i}$  is  $S_{k_1, k_2}$ -decomposable, then  $G$  is  $S_{k_1, k_2}$ -decomposable too, for every double-star  $S_{k_1, k_2}$  of size  $k$ . In [3, Theorem 3], the following was proved.

**Theorem 4.** *Let  $C_i : v_1, e_1, \dots, v_t, e_t, v_1$ , where  $1 \leq i \leq l$ . Then  $N_{G \setminus F}^+(v_j)$  can be partitioned into  $X_j$  and  $Y_{j-1}$  such that  $|X_j| = k_1$ ,  $|Y_{j-1}| = k_2$  and  $G_{C_i}[\{v_j, v_{j+1}\} \cup X_j \cup Y_j]$  is isomorphic to  $S_{k_1, k_2}$ , for  $j = 1, \dots, t$ .*

As a straightforward result of Theorem 4, we prove the following corollary.

**Corollary 3.** *Let  $G$  be a  $(2k + 1)$ -regular graph and  $k_1$  and  $k_2$  be two positive integers such that  $k_1 + k_2 = k$ . If  $G$  has a 2-factor, then  $G$  is  $S$ -decomposable, for  $S = \{S_{k_1, k_2}, S_{k_1-1, k_2}, S_{k_1, k_2-1}, S_{k_1-1, k_2-1}\}$ .*

*Proof.* Let  $H = G \square K_2$ . Now,  $H$  is a  $(2k + 2)$ -regular graph. Let  $F_1$  and  $F_2$  be two 2-factors in two copies of  $G$ , namely  $G_1$  and  $G_2$ . Clearly,  $F_1 \cup F_2$  is a 2-factor in  $H$ . Now, Theorem 4 implies that  $H$  has an  $S_{k_1, k_2}$  decomposition in which the central edges of double stars are exactly the edges of  $F_1 \cup F_2$ . Clearly, in this decomposition each double-star has at most two edges with one end point in  $V(G_1)$  and other end point in  $V(G_2)$ . By restriction of this decomposition to  $G_1$ , we are done.  $\square$

Here, we prove two results about the double-star decomposition of  $(2k + 1)$ -regular graphs.

**Theorem 5.** *Let  $G$  be a  $(2k + 1)$ -regular graph containing two disjoint perfect matchings  $M_1$  and  $M_2$ . If  $k_1$  and  $k_2$  are two positive integers such that  $k_1 + k_2 = k$ , then  $G$  is  $\{S_{k_1, k_2}, S_{k_1-1, k_2}\}$ -decomposable.*

*Proof.* Since  $G \setminus M_1$  is a  $2k$ -regular graph, it has an Eulerian orientation. Consider an orientation for  $M_1$ . So,  $G$  has a  $\{k, k + 1\}$ -orientation, say  $O$ , where  $M_1$  is a perfect matching between vertices of out-degree  $k$  and vertices of out-degree  $k + 1$ . Let  $H = G \square K_2$  and  $G_1$  and  $G_2$  be two copies of  $G$  in  $H$  with  $V(G_1) = \{v_1, \dots, v_n\}$  and  $V(G_2) = \{v'_1, \dots, v'_n\}$ . Also, suppose that  $E'$  is the set of all edges between  $G_1$  and  $G_2$  i.e.  $E' = \{v_i v'_i | i = 1, \dots, n\}$ . Clearly,  $H$  is  $(2k + 2)$ -regular. Consider orientations  $O$  and  $O'$  for  $G_1$  and  $G_2$ , respectively, where orientation of  $O'$  is reverse of  $O$ . Also, orient the edges of  $E'$  from the vertices of out-degree  $k$  to the vertices of out-degree  $k + 1$ . This is an Eulerian orientation for  $H$ . Let  $F_1 = M_1 \cup M_2$  and  $F_2 = M'_1 \cup M'_2$  in which  $M'_i$  is corresponding perfect matching in  $G_2$ ,

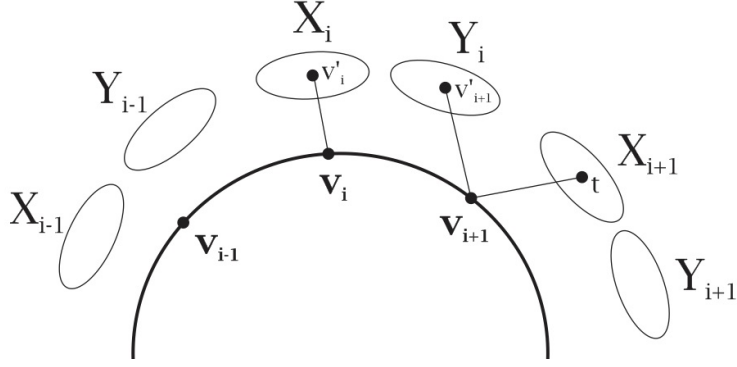


Figure 1.

for  $i = 1, 2$ . Now,  $F = F_1 \cup F_2$  is a 2-factor of  $H$  and Theorem 4 implies that  $H$  can be decomposed into  $S_{k_1, k_2}$  such that all the central edges of double-stars are the edges of  $F$ . Now, we prove two following claims.

**Claim 1.** The graph  $H$  has an  $S_{k_1, k_2}$ -decomposition such that each double-star in  $G_1$  has at most one edge in  $E'$ .

Let  $C : v_1, e_1, v_2, \dots, v_l, e_l, v_1$  be a cycle in  $F_1$ . Suppose that the double-star  $S_{e_i}$ , corresponding to  $e_i$ , has two edges in  $E'$ . This implies that  $d_{G_1}^+(v_i) = d_{G_1}^+(v_{i+1}) = k$  and  $e_i \in M_2$ . Thus  $e_{i-1}, e_{i+1} \in M_1$  and hence  $d_{G_1}^+(v_{i-1}) = d_{G_1}^+(v_{i+2}) = k + 1$ . For every  $v_j \in V(C)$ , let  $X_j$  be the set of all pendant vertices adjacent to  $v_j$  in  $S_{e_j}$ . Similarly, let  $Y_j$  be the set of all pendant vertices adjacent to  $v_{j+1}$  in  $S_{e_j}$ . With no loss of generality assume that  $|X_j| = k_1$  and  $|Y_j| = k_2$ , for  $j = 1, \dots, l$ . Since  $S_{e_i}$  has two edges in  $E'$ , we have  $v'_i \in X_i$  and  $v'_{i+1} \in Y_i$ . We have  $v'_i \notin X_{i-1}$  and  $v'_{i+1} \notin Y_{i+1}$ .

Note that since  $|X_i| = |X_{i+1}| = k_1$  and  $v'_i \notin X_{i+1}$ , we conclude that there exists  $t \in X_{i+1} \setminus X_i$ . Now, define  $X'_{i+1} = (X_{i+1} \setminus \{t\}) \cup \{v'_{i+1}\}$  and  $Y'_i = (Y_i \setminus \{v'_{i+1}\}) \cup \{t\}$ , see Figure 1. Let  $S'_{e_{i+1}}$  be the double-star with pendant sets  $X'_{i+1}$  and  $Y_{i+1}$ . Similarly, let  $S'_{e_i}$  be the double-star with pendant sets  $X_i$  and  $Y'_i$ . For any  $j \neq i, i + 1$ , let  $S'_{e_j} = S_{e_j}$ .

By repeating this procedure, one can obtain an  $S_{k_1, k_2}$ -decomposition for  $H$  in which every double-star has at most one edge in  $E'$ . This completes the proof of Claim 1.

**Claim 2.** There exists an  $S_{k_1, k_2}$ -decomposition for  $H$  in which every edge of each double-star contained in  $E'$  is incident with a vertex of degree  $k_1$ .

Consider the decomposition given in the Claim 1. Suppose that  $v'_{i+1} \in Y_i$ . With no loss of generality, we may assume that  $v'_2 \in Y_1$ . Thus,  $v'_2 \notin Y_2 \cup Y_l$ . This implies that  $Y_1 \notin \{Y_2, Y_l\}$ . Now, we consider two cases:

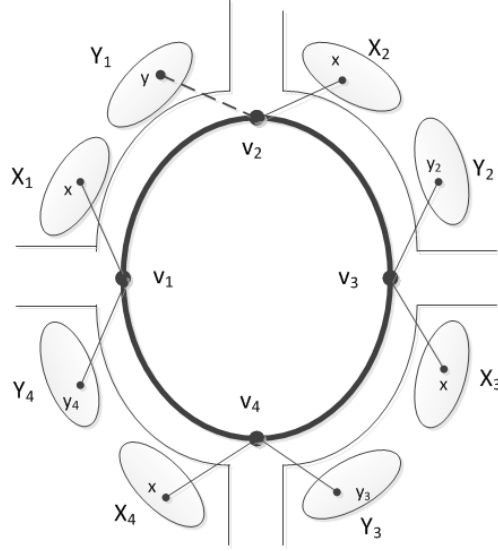


Figure 2.

**Case 1.**  $X_1 \neq X_2$ .

There exists  $x \in X_2 \setminus X_1$ . Now, define  $Y'_1 = (Y_1 \setminus \{v'_2\}) \cup \{x\}$  and  $X'_2 = (X_2 \setminus \{x\}) \cup \{v'_2\}$ . Let  $S'_{e_1}$  be the double-star with pendant sets  $X_1$  and  $Y'_1$  and  $S'_{e_2}$  be the double-star with pendant sets  $X'_2$  and  $Y_2$ . Now, by considering  $S'_{e_j} = S_{e_j}$ , for  $j \in \{3, 4, \dots, l\}$ , the result follows.

**Case 2.**  $X_1 = X_2$ . We consider two subcases:

**Subcase 2.1.**  $X_1 = X_2 = \dots = X_l$ .

Let  $x \in X_1$ . Since  $Y_1 \neq Y_l$ , there exists  $y_l \in Y_l \setminus Y_1$ . Now, let  $X'_1 = (X_1 \setminus \{x\}) \cup \{y_l\}$  and  $Y'_l = (Y_l \setminus \{y_l\}) \cup \{x\}$ . Since  $X_1 = X_l$  and  $x \in Y'_l$ , we conclude that  $x \notin Y_{l-1}$  and hence there exists  $y_{l-1} \in Y_{l-1} \setminus Y'_l$ . Now, let  $X'_l = (X_l \setminus \{x\}) \cup \{y_{l-1}\}$  and  $Y'_{l-1} = (Y_{l-1} \setminus \{y_{l-1}\}) \cup \{x\}$ . Repeat this procedure  $l$  times. In the final step, let  $X'_2 = (X_2 \setminus \{x\}) \cup \{v'_2\}$  and  $Y'_1 = (Y_1 \setminus \{v'_2\}) \cup \{x\}$ . One can see that  $X'_i \cap Y'_i = \emptyset$ , for  $i = 1, 2, \dots, l$ . Let  $S'_{e_i}$  be the double-star with pendant sets  $X'_i$  and  $Y'_i$ , see Figure 2. By repeating this procedure we obtain the desired decomposition.

**Subcase 2.2.**  $X_1 \neq X_t$ , for some  $3 \leq t \leq l$ .

Without loss of generality, we may assume that  $X_1 = X_l = \dots = X_{t+1} \neq X_t$ . Hence there exists  $x \in X_1 \setminus X_t$ . Similar to the previous subcase, we can define  $X'_1, X'_l, \dots, X'_{t+1}$  and  $Y'_l, Y'_{l-1}, \dots, Y'_t$ . We have  $X'_1 \neq X_2$  and according to the Case 1, we are done.

Now, these two claims yield that the restriction of decomposition of  $H$  to  $G_1$  is an  $\{S_{k_1, k_2}, S_{k_1-1, k_2}\}$ -decomposition. This completes the proof.  $\square$

In the following theorem, we find another result about the double-star decomposition of  $(2k+1)$ -regular graphs.

**Theorem 6.** *Let  $G$  be a  $(2k+1)$ -regular graph containing a perfect matching  $M$  and  $F$  be a 2-factor in  $G \setminus M$ . Suppose that  $k_1$  and  $k_2$  are two positive integers such that  $k_1 + k_2 = k - 1$ . If for every two adjacent vertices  $u$  and  $v$  in  $F$ ,  $|N_G(u) \cap N_G(v)| \leq k_1 - 1$ , then  $G$  is  $\{S_{k_1, k_2}, S_{k_1+1, k_2}\}$ -decomposable.*

*Proof.* Theorem 4 implies that  $G \setminus M$  can be decomposed into  $S_{k_1, k_2}$  such that the edges of  $F$  are the central edges of double-stars. Now, let  $e = uv \in M$  and  $C : v_1, e_1, v_2, \dots, v_t, e_t, v_1$  be a cycle in  $F$  containing  $u$ . Without loss of generality we may assume that  $u = v_1$ . Suppose that  $X_i$  is the set of all pendant vertices adjacent to  $v_i$  in  $S_{e_i}$ , for  $i = 1, 2, \dots, t$ . Similarly, let  $Y_i$  be the set of all pendant vertices adjacent to  $v_{i+1}$  in  $S_{e_i}$ . With no loss of generality assume that  $|X_i| = k_1$  and  $|Y_i| = k_2$ . We consider two cases:

**Case 1.**  $v \notin Y_1$ .

Let  $X'_1 = X_1 \cup \{v\}$ . Let  $S'_{e_1}$  be the double-star with pendant sets  $X'_1$  and  $Y_1$ . Also, let  $S'_e = S_e$ , for  $e \neq e_1$ . Clearly, every double-star is  $S_{k_1, k_2}$  or  $S_{k_1+1, k_2}$ .

**Case 2.**  $v \in Y_1$ .

Note that in this case,  $v \notin X_2$ . Since  $|N_G(v_1) \cap N_G(v_2)| \leq k_1 - 1$ , there exists  $x \in X_2 \setminus X_1$ . Let  $X'_2 = (X_2 \setminus \{x\}) \cup \{v\}$  and  $Y'_1 = (Y_1 \setminus \{v\}) \cup \{x\}$  and for  $i \neq 2$  define  $X'_i = X_i$  and for  $j \geq 2$ ,  $Y'_j = Y_j$ .

Now,  $v \in X'_2$ . If  $v \notin Y'_2$ , then we are done. If  $v \in Y'_2$ , then repeat this procedure till  $v \in X'_j \setminus Y'_j$ . Note that this can be done since  $v \notin Y_t$ . Let  $S'_{e_i}$  be the double-star with pendant sets  $X'_i$  and  $Y'_i$  for  $e_i \in E(C)$  and  $S'_{e_j} = S_{e_j}$  for  $e_j \notin E(C)$ . Obviously, every double-star is  $S_{k_1, k_2}$  or  $S_{k_1+1, k_2}$ .

For every  $e = uv \in M$ , there exists a unique double-star in  $G \setminus M$  in which  $u$  (or  $v$ ) is a support vertex of degree  $k_1$ . These imply that  $G$  is  $\{S_{k_1, k_2}, S_{k_1+1, k_2}\}$ -decomposable.  $\square$

Now, the following corollary is clear.

**Corollary 4.** *Let  $G$  be a triangle-free  $(2k+1)$ -regular graph containing a perfect matching*



and  $k_1, k_2$  be two positive integers such that  $k_1 + k_2 = k - 1$ . Then  $G$  is  $\{S_{k_1, k_2}, S_{k_1+1, k_2}\}$ -decomposable.

We have an immediate corollary.

**Corollary 5.** *Let  $r$  and  $k$  be two positive integers such that  $k \mid r$ . Suppose that  $S$  is a double-star of size  $k$ . If  $G$  is a  $2r$ -regular graph, then  $G$  is decomposed into  $S$ .*

*Proof.* Notice that since  $k \mid r$ , it is straight forward to see that  $G$  can be decomposed into  $G_1, \dots, G_s$  such that each  $G_i$  is spanning and  $2k$ -regular for  $i = 1, \dots, s$ . Now, Theorem 1 yields that each  $G_i$  is  $S$ -decomposable and this completes the proof.  $\square$

**Corollary 6.** *Let  $G$  be a 1-factorable  $(2k + 1)$ -regular graph. If  $r \geq 3$  is a positive integer such that  $r \mid 2k + 1$ , then  $G$  is  $\{S_{k_1, k_2}, S_{k_1-1, k_2}\}$ -decomposable, for any double-star  $S_{k_1, k_2}$  of size  $r$ .*

*Proof.* Let  $2k + 1 = rs$ . It is easy to see that  $G$  can be decomposed into spanning subgraphs  $G_1, \dots, G_s$  such that each  $G_i$  is  $r$ -regular, for  $i = 1, \dots, s$ . Now, Theorem 5 completes the proof.  $\square$

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